Simultaneous Material/Load/Shape Variations of Thermoelastic Structures

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Simultaneous variations of material properties, load functions, and shape configurations of dynamic thermoelastic structures are considered. A continuous approach is adopted in order to find the total variation of a general performance criterion. The present analysis may find important physical applications in the simultaneous shape optimization and control of large space structures in time by applied thermal and/or mechanical loads. The adjoint variable method and the material derivative concept are used to find the sensitivity expressions, which are checked by a simple one-dimensional example problem.

Nomenclature

a = heat source decision parameter
a_k = material/load decision parameters
B = body force in example problem
b = body force decision parameter
b_i = body forces
c = heat capacity
c_0 = wave speed in example problem
E = modulus of elasticity
f,g,h = integrands of performance criterion
H = boundary curvature
I = general performance criterion
$\tilde{I}$ = augmented performance criterion
k = thermal conductivity
L = length of rod
n_i = unit vector normal to boundary surface
Q = distributed heat source
q = normal boundary heat flux
q^* = adjoint normal boundary heat flux
q_i = heat-flow vector
q_i^* = adjoint heat-flow vector
T = interval of time, $[0,t_f]$
t = time variable
t_f = final time
t_i = boundary tractions
t_i^* = adjoint boundary tractions
u = x component of displacements
u^* = adjoint displacement in x direction
u_i = displacements
u_i^* = adjoint displacements
V_i = deformation velocity vector
V_n = normal component of V_i
V_μ = component of V_i normal to Λ and tangent to Γ
w = dummy variable
x = axial direction in rod
x_i = Cartesian coordinates
α = thermal diffusivity
β = linear coefficient of thermal expansion
γ = material property given by $(3\lambda+2\mu)\beta$
Γ = boundary surfaces
$\Gamma_\theta,\Gamma_q,\Gamma_u,\Gamma_t$ = boundary segments
δ = variation symbol
δ_{ij} = Kronecker delta
ϵ_{ij} = strain tensor
ζ_0 = fundamental characteristic value, $= \pi/2L$
θ = temperature
θ_r = reference temperature
θ^* = adjoint temperature
λ = Lame's constant
Λ = discontinuity surface line on Γ
μ = Lame's constant
ρ = density
σ = normal stress in x direction
σ^* = adjoint normal stress in x direction
σ_{ij} = stress tensor
σ_{ij}^* = adjoint stress tensor
τ = pseudotime variable (for shape variations)
ϕ = general material/load decision variable
Ψ_1,Ψ_2 = general integrals
Ω = physical domain

Subscripts
$,i$ = derivative with respect to x_i
$,n$ = normal derivative
0 = initial value

Superscripts
$(\ )^0$ = boundary value
$(\ )'$ = local derivative
$(\dot{\ })$ = time derivative

Introduction

SIMULTANEOUS variations of material properties, distributed and boundary loads or initial conditions (henceforth called "load," in general), and also of shape configurations may be very important from physical, economical, etc., points of view in optimization and/or control of structures in dynamic response. For example, in the design of large space structures, it is of crucial importance that they are of light weight, because of the high cost of lifting mass to orbit, and that they satisfy very stringent requirements on their shape accuracy.[1-5] In this respect, considerable interest has been generated recently in the joint minimization of structural parameters and control variables. To maintain a desired shape with necessary accuracy under given disturbances, passive and active control systems have been proposed.[1-5] In Ref. 1, applied heating and applied forces have been suggested in order to nullify some distortions in the shape of structures to control the quasisteady types of deformations. Optimum design of structures under mechanical and thermal loads has also been considered in Ref. 6 under steady-state conditions.

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In all of the previous references, variations are performed on discrete approximations of structures. Hence, the system's equations are given only by ordinary differential equations in the time variable. Variation of material properties and shape optimization of continuous elastic structures under steady-state conditions have been developed rather thoroughly.[7-9] One of the straightforward methods of structural optimization for continuous structures is derived from the material derivative concept.[7] The material derivative formulation of shape variations has been extended recently for linearly elastic structures under dynamic loads for time-independent domains.[10] The present paper further generalizes the method by treating the joint material/load/shape (MLS) variations of thermoelastic structures in dynamic response. As such, the derived sensitivity analysis (SA) expressions should be of great use for simultaneous shape optimization and dynamical control of structures under thermomechanical loads. A very recent parallel study was also performed by Dems and Mroz,[11] in which they formulated separate SA expressions for material and shape variations by direct and adjoint variable methods of optimization. In the present paper, though, the simultaneous SA results are presented and checked by a one-dimensional example problem considering dynamic effects.

Problem Definition

Neglecting thermomechanical coupling effects,[11] the field equations of linear thermoelasticity in dynamic response may be written as follows[12]:

in Ω :

$$-q_{i,i} + Q = \rho c \dot{\theta} \tag{1}$$

$$\sigma_{ij,j} + b_i = \rho \ddot{u}_i \tag{2}$$

where Ω is the time-independent physical domain. Index notation, with summations indicated by repeated indices, is employed.

For the field equations, Eqs. (1) and (2), relevant boundary conditions may be imposed in the following form:

on Γ_θ:

$$\theta = \theta^0 \tag{3}$$

on Γ_q:

$$q = q_i n_i = q^0 \tag{4}$$

on Γ_u:

$$u_i = u_i^0 \tag{5}$$

on Γ_t :

$$t_i = \sigma_{ij} n_j = t_i^0 \tag{6}$$

where the boundary pair Γ_θ and Γ_q are disjoint sets, as well as the pair Γ_u and Γ_t, with $\Gamma = \Gamma_\theta U \Gamma_q = \Gamma_u U \Gamma_t$.

As the field equations constitute an initial boundary-value problem, initial conditions for the thermomechanical system also must be provided, which may be written as follows:

at $t = 0$:

$$\theta = \theta_0 \tag{7}$$

$$u_i = u_{i0} \tag{8}$$

$$\dot{u}_i = \dot{u}_{i0} \tag{9}$$

It is noted that the boundary conditions hold for $t > 0$, whereas the initial conditions are valid within the whole physical domain at $t = 0$.

The Fourier law of heat condition

$$q_i = -k\theta_{,i} \tag{10}$$

and the Duhamel-Neuman conditions for the stress-strain relationship

$$\theta_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} - \gamma(\theta - \theta_r)\delta_{ij} \tag{11}$$

are taken as the constitutive equations for the present thermomechanical structure. The strain tensor is related to the displacement components by the kinematic conditions as given by

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{12}$$

It is assumed that we are interested in the system's response within a finite time interval $T = [0,t_f]$. Within the time interval T, the following so-called decision variables will be varied as follows: 1) material properties: k, λ, μ, ρ, c, and β; 2) loading functions: Distributed loads—Q and b_i; Boundary loads—θ^0, q^0, u_i^0, and t_i^0; Initial loads—θ_0, u_{i_0}, and $\dot{u}_{i_0}$; 3) shape configuration: Ω (and, hence, Γ).

For the simultaneous MLS sensitivity analysis, a general integral functional I is considered. This functional, termed as the general performance criterion, is defined on the space-time domain $\Omega \times T$, and may serve as the objective constraint function to be satisfied in an optimization and/or control problem. That is,

$$I = \int_T \int_\Omega f(\theta, q_i, u_i, \dot{u}_i, \sigma_{ij}, \epsilon_{ij}, Q, k, b_i, \lambda, \mu, \rho, c, \beta)\, d\Omega\, dt$$
$$+ \int_T \int_\Gamma g(\theta, q, u_i, t_i)\, d\Gamma\, dt + \int_\Omega [h(\theta, u_i, \dot{u}_i)]_T\, d\Omega \tag{13}$$

where the symbol $[\ ]_T$ denotes $[\ ]_{t=t_f} - [\ ]_{t=0}$, and f, g, and h are continuous and differentiable functions with respect to their arguments. It may be noted that the functional I is more general than the one treated in Ref. 11 in that it also involves stresses, heat flows, and velocities in the space-time domain, as well as loading functions for control problems.

Total Derivatives

The material derivative concept,[7] which has been extended for dynamic systems[9] in shape variation problems, is now developed further for the present joint MLS variation of functions and integral functionals. Following Ref. 7, the variation of Ω under a shape transformation, characterized by a pseudotime parameter τ, may be regarded as a "dynamic" deformation of a continuous medium. Thus, a point x_i in Ω (at $\tau = 0$) moves to the point x_i^τ in the varied domain Ω^τ, given by

$$x_i^\tau = x_i + \tau V_i(x_j) \tag{14}$$

where the deformation "velocity" field V_i is time independent and is defined in the whole physical space, representing the rate of deformation (i.e., variation) in a shape variation situation.

Suppose that the system state (i.e., θ and u_i for the present thermoelastic solid) depends on a material/load function, denoted by ϕ, which may be a function of the space and time variables, the shape configuration, and a set of material/load (constant) parameters a_k as follows:

$$\phi = \phi(x_i, t, a_k) \tag{15}$$

The function ϕ may represent any of the material or load decision functions, e.g., k, θ^0, b_i, t_i^0, etc. It is noted that ϕ is taken as dependent on the material points of the structure for generality. In particular, for the case of pressure loading on Γ_t, the surface tractions t_i^0 may be given by $t_i^0 = p^0 n_i$, where p^0 is the pressure, thus depending on the normal to Γ_t (or the space configuration).

Any shape variation of the domain, characterized by Eq. (14), may result in the varied function ϕ^τ, given by

$$\phi^\tau(x_i^\tau, t, a_k) = \phi^\tau(x_i + \tau V_i, t, a_k) \tag{16}$$

where t and a_k have been held constant. If simultaneous MLS variations are considered, the local variation in ϕ, indicated by ϕ', may be given as

$$\phi' = \frac{\partial \phi}{\partial a_k}\,\delta a_k + \lim_{\tau\to 0}\frac{\phi^\tau(x_i,t,a_k)-\phi(x_i,t,a_k)}{\tau} \tag{17}$$

where the first variations in a_k are denoted by δa_k. In Eq. (17), the first term on the right-hand side represents the contribution to ϕ' of the material/load parameter variations, whereas the second term results from shape variations, holding x_i, t, and a_k constant. If ϕ depends only on space points (but not material points), any shape variation will not have any effect on ϕ, leading to $\phi^\tau = \phi$. Thus, the last term in Eq. (17) would be identically equal to zero.

A general continuous function w, on the other hand, may depend on the material/load function ϕ, as well as the material points and the time as follows:

$$w = w(x_i, t, \phi) \tag{18}$$

Any shape variation of the domain may result in the varied function w^τ, given by

$$w^\tau(x_i^\tau, t, \phi^\tau) = w^\tau[x_i + \tau V_i, t, \phi^\tau(x_i + \tau V_i, t, a_k)] \tag{19}$$

In all of the preceding equations, all of the functions are assumed to depend on their arguments continuously. For a simultaneous MLS variation, the local variation in w may be written in the following form by holding x_i and t constant:

$$w' = \frac{\partial w}{\partial \phi}\phi' + \lim_{\tau\to 0}\frac{w^\tau(x_i,t,\phi)-w(x_i,t,\phi)}{\tau} \tag{20}$$

where ϕ' is given by Eq. (17).

When simultaneous variations are considered, the total derivation by w, denoted by $\mathrm{D}w$, can be expressed as the sum of the local derivative of w, i.e., w', holding x_i and t constant, and the variation in w when x_i is moved with the deformation velocity V_i as follows:

$$\mathrm{D}w = w' + w_{,i}V_i \tag{21}$$

where, noting Eqs. (15) and (18), $\mathrm{D}w$ is defined as

$$\mathrm{D}w = \frac{\partial w}{\partial \phi}\frac{\partial \phi}{\partial a_k}\,\delta a_k + \lim_{\tau\to 0}\frac{w^\tau(x_i^\tau,t,\phi^\tau)-w(x_i,t,\phi)}{\tau} \tag{22}$$

It may be observed from Eq. (22) that the first term on the right-hand side comes from the variations in the material/load parameters a_k, whereas the second term results from shape variations, due to the explicit and implicit dependence of w on shape through x_i and ϕ, respectively.

For the simultaneous MLS sensitivity analysis of the general performance criterion I, the total derivatives of integral functionals defined on $\Omega \times T$ and $\Gamma \times T$ will be needed. Extending the material derivative formulas of integrals for shape variations under static response of functions,[7] the total derivative of space-time integrals with respect to simultaneous MLS variations may be generated directly. In this respect, a general integral in $\Omega \times T$ may be defined in terms of a differentiable function $w(x_i, t, \phi)$ as follows:

$$\Psi_1 = \int_T \int_\Omega w \,\mathrm{d}\Omega\,\mathrm{d}t \tag{23}$$

Adopting the boundary method [10,13] of shape variations, the total derivative of Ψ_1 may be given in the following form:

$$\mathrm{D}\Psi_1 = \int_T \int_\Omega w' \,\mathrm{d}\Omega\,\mathrm{d}t + \int_T \int_\Gamma wV_n \,\mathrm{d}\Gamma\,\mathrm{d}t \tag{24}$$

where V_n is the normal component of V_i on Γ and is time independent; w' is given by Eqs. (17) and (20). For Eq. (24), it also has been assumed that no discontinuities exist in the functions w and V_i in the domain Ω, since integration by parts with no jump terms has been employed to derive the equation.

A general integral also may be defined over $\Gamma \times T$ for smooth surfaces as follows:

$$\Psi_2 = \int_T \int_\Gamma w \,\mathrm{d}\Gamma\,\mathrm{d}t \tag{25}$$

The total derivative of Ψ_2 is given in the following form[9,10]:

$$\mathrm{D}\Psi_2 = \int_T \int_\Gamma [w' + (w_{,n} + Hw)V_n]\,\mathrm{d}\Gamma\,\mathrm{d}t + \int_T \oint_\Lambda [\![w]\!]\, V_\mu \,\mathrm{d}\Lambda\,\mathrm{d}t \tag{26}$$

where $w_{,n}$ is the normal derivative of w on Γ, and H is the curvature of the boundary Γ in R^2 and twice the mean surface curvature of Γ in R^3. Any discontinuity of w across the boundary surface curve $\Lambda \in \Gamma$ is taken care of by the last integral over $\Lambda \times T$ in Eq. (26), where V_μ is the component of V_i on Γ, normal to Λ and tangent to Γ[14]; the quantity $[\![w]\!]$ indicates the jump across the (boundary data) discontinuity curve Λ, i.e., $[\![w]\!] = w^- - w^+$, where w^- and w^+ are the values of w immediately near Λ at the negative and positive sides, respectively. The line integral over Λ, for example, can be selected along the boundary curve separating the boundary portions S_u and S_t, thus describing the interaction between boundaries with discontinuous boundary conditions. For nonsmooth boundary surfaces, the variations of V_μ over the geometric discontinuity lines also must be considered. For example, for a boundary composed of a set of regular surfaces intersecting at the edges, summations must be taken over the discontinuity lines,[14] differentiating between $V_{\mu+}$ and $V_{\mu-}$ across the edges, as they would not be colinear as in the case of smooth boundaries.

MLS Sensitivity Analysis

As the total derivatives of functions and integral functionals have been developed for MLS variations in the preceding section, the SA of the general performance criterion I, Eq. (13), is now formulated with respect to the MLS decision variables. Before proceeding with the SA, however, it may be worthwhile to point out the applications and importance of simultaneous MLS sensitivity analysis of physical systems that are under dynamic loads, namely: 1) to understand and model system's behavior better, 2) to optimize the system's response and/or physical shapes in a prescribed time interval, 3) to control the system's responses in time for desired objectives (i.e., open- or closed-loop controls of loading functions), or 4) for identification of MLS variables by using the system's measured responses in time. The present SA problem now may be stated as follows. Find the total derivatives of I with respect to MLS variations subject to the primary problem defined by Eqs. (1-12). The SA expressions, which should be derived for each integral functional present in an optimization or control problem, for example, may be used effectively in the mathematical programming methods of minimizing a functional subject to nonlinear constraints.

The first step in the SA by the adjoint variable method may be the definition of an augmented functional $\tilde{I}$ by incorporating the field equations (1) and (2) into I, i.e.,

$$\tilde{I} = I + \int_T \int_\Omega [\theta^*(-q_{i,i} + Q - \rho c\dot{\theta}) + u_i^*(\sigma_{ij,j} + b_i - \rho\ddot{u}_i)] \, d\Omega \, dt \tag{27}$$

The original performance criterion I, Eq. (13), may be inserted into Eq. (27) and integration by parts in Ω and T may be employed in order to reduce the order of differentiations leading to the following expressions:

$$\tilde{I} = \int_T \int_\Omega [f + q_i\theta^*_{,i} + (Q - \rho c\dot{\theta})\theta^* - \sigma_{ij}u^*_{i,j} + b_i u_i^* + \rho\dot{u}_i\dot{u}_i^*] \, d\Omega \, dt + \int_T \int_\Gamma (g - q\theta^* + t_i u_i^*) \, d\Gamma \, dt + \int_\Omega [h - \rho\dot{u}_i u_i^*]_T \, d\Omega \tag{28}$$

The total derivative of $\tilde{I}$, Eq. (28), corresponding to simultaneous MLS variations of the decision variables may be taken by using the general "total derivative formulas," Eqs. (23-26), as follows:

$$\begin{aligned}
D\tilde{I} = {} & \int_T \int_\Omega \Bigg[\frac{\partial f}{\partial \theta}\theta' + \left(\frac{\partial f}{\partial_{q_i}} + \theta^*_{,i}\right)q_i' + \frac{\partial f}{\partial u_i}u_i' \\
& + \left(\frac{\partial f}{\partial \dot{u}_i} + \rho\dot{u}_i^*\right)\dot{u}_i' + \frac{\partial f}{\partial \epsilon_{ij}}u'_{i,j} + \left(\frac{\partial f}{\partial \sigma_{ij}} - u^*_{i,j}\right)\sigma'_{ij} + q_i\theta^{*\prime}_{,i} \\
& + (Q - \rho c\dot{\theta})\theta^{*\prime} - \rho c\theta^*\dot{\theta}' - \sigma_{ij}u^{*\prime}_{i,j} + b_i u_i^{*\prime} + \rho\dot{u}_i\dot{u}_i^{*\prime} \\
& + \left(\frac{\partial f}{\partial Q} + \theta^*\right)Q' + \frac{\partial f}{\partial k}k' + \left(\frac{\partial f}{\partial b_i} + u_i^*\right)b_i' + \frac{\partial f}{\partial \lambda}\lambda' \\
& + \frac{\partial f}{\partial \mu}\mu' + \frac{\partial f}{\partial \beta}\beta' + \left(\frac{\partial f}{\partial c} - \rho\dot{\theta}\theta^*\right)c' \\
& + \left(\frac{\partial f}{\partial \rho} - c\dot{\theta}\theta^* + \dot{u}_i\dot{u}_i^*\right)\rho' \Bigg] \, d\Omega \, dt \\
& + \int_T \int_\Gamma \Big\{ [f + q_i\theta^*_{,i} + (Q - \rho c\dot{\theta})\theta^* - \sigma_{ij}u^*_{i,j} \\
& + b_i u_i^* + \rho\dot{u}_i\dot{u}_i^* + (g - q\theta^* + t_i u_i^*)_{,n} \\
& + H(g - q\theta^* + t_i u_i^*)]V_n + \frac{\partial g}{\partial \theta}\theta' + \left(\frac{\partial g}{\partial q} - \theta^*\right)q' \\
& + \frac{\partial g}{\partial u_i}u_i' + \left(\frac{\partial g}{\partial t_i} + u_i^*\right)t_i' - q\theta^{*\prime} + t_i u_i^{*\prime} \Big\} \, d\Gamma \, dt \\
& + \int_\Omega \Bigg[\frac{\partial h}{\partial \theta}\theta' + \frac{\partial h}{\partial u_i}u_i' + \left(\frac{\partial h}{\partial \dot{u}_i} - \rho u_i^*\right)\dot{u}_i' \\
& - \rho\dot{u}_i u_i^{*\prime} - \dot{u}_i u_i^*\rho' \Bigg]_T \, d\Omega + \int_\Gamma [h - \rho\dot{u}_i u_i^*]_T V_n \, d\Gamma \\
& + \int_T \oint_\Lambda [\![g - q\theta^* + t_i u_i^*]\!] V_\mu \, d\Lambda \, dt
\end{aligned} \tag{29}$$

where the boundary date discontinuity curve $\Lambda \in \Gamma$ now represents the sum of the intersection boundary curves between Γ_θ and Γ_q, and between Γ_u and Γ_t.

The local derivative forms (incorporating the effects of MLS variations at constant x_i and t) of the constitutive and kinematic equations, Eqs. (10-12), are simply given as follows:

$$q_i' = -k\theta_{,i}' - \theta_{,i}k' \tag{30}$$

$$\sigma_{ij}' = [\lambda u'_{k,k} + u_{k,k}\lambda' - \tau\theta' - \beta\theta(3\lambda' + 2\mu') - (3\lambda + 2\mu)\theta\beta']\delta_{ij} + \mu(u'_{i,j} + u'_{j,i}) + (u_{i,j} + u_{j,i})\mu' \tag{31}$$

$$\epsilon_{ij}' = \tfrac{1}{2}(u'_{i,j} + u'_{j,i}) \tag{32}$$

Introducing Eqs. (30-32) into Eq. (29) and using integration by parts in Ω and T repeatedly, the total derivative $D\tilde{I}$ may be written in the following form:

$$\begin{aligned}
D\tilde{I} = {} & \int_T \int_\Omega \Big\{ (-q_{i,i} + Q - \rho c\dot{\theta})\theta^{*\prime} + (\sigma_{ij,j} + b_i - \rho\ddot{u}_i)u_i^{*\prime} \\
& + \left[-q^*_{i,i} + \frac{\partial f}{\partial \theta} + \gamma\left(u^*_{k,k} - \frac{\partial f}{\partial \sigma_{kk}}\right) + \rho c\dot{\theta}^* \right]\theta' \\
& + \left[\sigma^*_{ij,j} + \frac{\partial f}{\partial u_i} - \left(\frac{\partial f}{\partial \dot{u}_i}\right)^{\cdot} - \rho\ddot{u}_i^* \right]u_i' + \left(\frac{\partial f}{\partial Q} + \theta^*\right)Q' \\
& + \left(\frac{\partial f}{\partial b_i} + u_i^*\right)b_i' + \left[\frac{\partial f}{\partial k} - \left(\theta^*_{,i} + \frac{\partial f}{\partial q_i}\right)\theta_{,i} \right]k' \\
& + \left[\frac{\partial f}{\partial \lambda} - (u_{i,i} - 3\beta\theta)\left(u^*_{j,j} - \frac{\partial f}{\partial \sigma_{jj}}\right) \right]\lambda' \\
& + \left[\frac{\partial f}{\partial \mu} - (u_{i,j} + u_{j,i} - 2\beta\theta\delta_{ij})\left(u^*_{i,j} - \frac{\partial f}{\partial \sigma_{ij}}\right) \right]\mu' \\
& + \left[\frac{\partial f}{\partial \beta} + (3\lambda + 2\mu)\left(u^*_{i,i} - \frac{\partial f}{\partial \sigma_{ii}}\right)\theta \right]\beta' \\
& + \left(\frac{\partial f}{\partial \rho} - c\dot{\theta}\theta^* + \dot{u}_i\dot{u}_i^*\right)\rho' + \left(\frac{\partial f}{\partial c} - \rho\dot{\theta}\theta^*\right)c' \Big\} \, d\Omega \, dt \\
& + \int_T \int_\Gamma \Big\{ [f + q_i\theta^*_{,i} + (Q - \rho c\dot{\theta})\theta^* - \sigma_{ij}u^*_{i,j} + b_i u_i^* \\
& + \rho\dot{u}_i\dot{u}_i^* + (g - q\theta + t_i u_i^*)_{,n} + H(g - q\theta^* + t_i u_i^*)]V_n \\
& + \left(\frac{\partial g}{\partial \theta} + q^*\right)\theta' + \left(\frac{\partial g}{\partial q} - \theta^*\right)q' + \left(\frac{\partial g}{\partial u_i} - t_i^*\right)u_i' \\
& + \left(\frac{\partial g}{\partial t_i} + u_i^*\right)t_i' \Big\} \, d\Gamma \, dt + \int_\Omega \Bigg[\left(\frac{\partial h}{\partial \theta} - \rho c\theta^*\right)\theta' \\
& + \left(\frac{\partial h}{\partial u_i} + \frac{\partial f}{\partial \dot{u}_i} + \rho\dot{u}_i^*\right)u_i' + \left(\frac{\partial h}{\partial \dot{u}_i} - \rho u_i^*\right)\dot{u}_i' - \dot{u}_i u_i^*\rho' \Bigg]_T \, d\Omega \\
& + \int_\Gamma [h - \rho\dot{u}_i u_i^*]_T V_n \, d\Gamma + \int_T \oint_\Lambda [\![g - q\theta^* + t_i u_i^*]\!] V_\mu \, d\Lambda \, dt
\end{aligned} \tag{33}$$

where the adjoint heat-flow vector q_i^* and the adjoint stress tensor σ^*_{ij} are defined by

$$q_i^* = -k\left(\theta^*_{,i} + \frac{\partial f}{\partial q_i}\right) \tag{34}$$

$$\sigma^*_{ij} = \lambda\left(u^*_{k,k} - \frac{\partial f}{\partial \sigma_{kk}}\right)\delta_{ij} + \mu\left[\left(u^*_{i,j} - \frac{\partial f}{\partial \sigma_{ij}}\right) + \left(u^*_{j,i} - \frac{\partial f}{\partial \sigma_{ji}}\right)\right] - \frac{\partial f}{\partial \epsilon_{ij}} \tag{35}$$

It may be noted that the constitutive equation relating σ^*_{ij} to the displacement gradients, Eq. (35), does not involve the adjoint temperature θ^*, whereas Eq. (11), governing the stress-strain relationship in the primary problem, features a θ dependency.

So far, the boundary and initial conditions, Eqs. (3-9), have not been employed during the SA procedure. The total derivative of Eq. (3), for example, leads to the following equation:

on Γ_θ:

$$D\theta = D\theta^0 \tag{36}$$

or, by using Eq. (21) for the left-hand side and rearranging terms,

on Γ_θ:

$$\theta' = D\theta^0 - \theta_{,k} V_k \tag{37}$$

where $D\theta^0$ is given by the total variation of the boundary temperature θ^0 on Γ_θ. Similarly, using the traction boundary condition on Γ_t, Eq. (6), the local derivative of t_i is given by

on Γ_t:

$$t_i' = Dt_i^0 - t_{i,k} V_k \tag{38}$$

Similar expressions may be derived for the other boundary and initial conditions.

It is emphasized that the total derivatives $D\theta^0$, Dt_i^0, etc., are known from the assumed forms of the boundary and initial conditions. In the case of conservative boundary loading on Γ_t, for example, the tractions are independent of the surface configuration, and the total derivative Dt_i^0 is simply given by

$$Dt_i^0 = t_i^{0'} + t_{i,k}^0 V_k \tag{39}$$

As an example of a nonconservative loading, on the other hand, consider the pressure loading

$$t_i^0 = p^0 n_i \tag{40}$$

directed along the normal to the surface. The total derivative of Eq. (40) simply leads to

$$Dt_i^0 = n_i \, Dp^0 + p^0 \, Dn_i \tag{41}$$

The total derivative of the unit normal vector n_i, Dn_i, is given by[8]

$$Dn_i = (n_i n_j - \delta_{ij}) n_k V_{k,j} \tag{42}$$

Using Eqs. (21), (41), and (42), Dt_i^0 finally can be expressed in the following form for the case of the pressure loading:

$$Dt_i^0 = (p^{0'} + p_{,k}^0 V_k) n_i + p^0 (n_i n_j - \delta_{ij}) n_k V_{k,j} \tag{43}$$

In the following, the explicit forms of the load decision variables θ, t_i^0, etc., will not be given; hence, their total derivatives must be calculated in accordance with their assumed forms in a specific situation.

The boundary surface Γ is decomposed into parts with different boundary conditions, cf., Eqs. (3-6), and the local derivative forms of the boundary and initial conditions are substituted into Eq. (33). It is then found suitable to set the coefficients of the local derivatives of the state functions such that an adjoint problem is defined as follows:

in Ω:

$$-q_{i,i}^* + \frac{\partial f}{\partial \theta} + \gamma \left(u_{i,i}^* - \frac{\partial f}{\partial \sigma_{ii}} \right) = -\rho c \dot{\theta}^* \tag{44}$$

$$\sigma_{ij,j}^* + \frac{\partial f}{\partial u_i} - \left(\frac{\partial f}{\partial \dot{u}_i} \right)^{\boldsymbol{\cdot}} = \rho \ddot{u}_i^* \tag{45}$$

on Γ_θ:

$$\theta^* = \frac{\partial g}{\partial q} \tag{46}$$

on Γ_q:

$$q^* = -\frac{\partial g}{\partial \theta} \tag{47}$$

on Γ_u:

$$u_i^* = -\frac{\partial g}{\partial t_i} \tag{48}$$

on Γ_t:

$$t_i^* = \frac{\partial g}{\partial u_i} \tag{49}$$

at $t = t_f$:

$$\rho c \theta^* = \frac{\partial h}{\partial \theta} \tag{50}$$

$$\rho u_i^* = \frac{\partial h}{\partial \dot{u}_i} \tag{51}$$

$$\rho \dot{u}_i^* = -\frac{\partial h}{\partial u_i} - \frac{\partial f}{\partial \dot{u}_i} \tag{52}$$

It may be noted that the adjoint "thermoelastic" problem is a final time-boundary-value problem, instead of an initial time-boundary-value problem, as in the case of the primary problem, cf., Eqs. (1-12). Furthermore, although the adjoint "heat conduction" equation (44) has a minus sign in front of its time-derivative term, $\dot{\theta}^*$, it retains its parabolic partial differential equation character.

When the primary and adjoint problems are satisfied for a given set of MLS decision variables, the total derivative of the general performance criterion I is finally expressed as follows:

$$\begin{aligned}
DI = & \int_T \int_\Omega \left\{ \left(\frac{\partial f}{\partial Q} + \theta^* \right) Q' + \left(\frac{\partial f}{\partial b_i} + u_i^* \right) b_i' \right. \\
& + \left[\frac{\partial f}{\partial k} - \theta_{,i} \left(\theta_{,i}^* + \frac{\partial f}{\partial q_i} \right) \right] k' + \left[\frac{\partial f}{\partial \lambda} - (u_{i,i} - 3\beta\theta) \right. \\
& \times \left. \left(u_{j,j}^* - \frac{\partial f}{\partial \sigma_{jj}} \right) \right] \lambda' + \left[\frac{\partial f}{\partial \mu} - (u_{i,j} + u_{j,i} - 2\beta\theta\delta_{ij}) \right. \\
& \times \left. \left(u_{i,j}^* - \frac{\partial f}{\partial \sigma_{ij}} \right) \right] \mu' + \left[\frac{\partial f}{\partial \beta} + (3\lambda + 2\mu) \left(u_{i,i}^* - \frac{\partial f}{\partial \sigma_{ii}} \right) \theta \right] \beta' \\
& \left. + \left(\frac{\partial f}{\partial \rho} - c\dot{\theta}\theta^* + \dot{u}_i \dot{u}_i^* \right) \rho' + \left(\frac{\partial f}{\partial c} - \rho \dot{\theta} \theta^* \right) c' \right\} d\Omega \, dt \\
& + \int_T \int_\Gamma [f + q_i \theta_{,i}^* + (Q - \rho c \dot{\theta}) \theta^* - \sigma_{ij} u_{i,j}^* + b_i + \rho \dot{u}_i \dot{u}_i^* \\
& + (g - q\theta^* + t_i u_i^*)_{,n} + H(g - q\theta^* + t_i u_i^*)] V_n \, d\Gamma \, dt \\
& + \int_T \int_{\Gamma_\theta} \left(\frac{\partial g}{\partial \theta} + q^* \right) (D\theta^0 - \theta_{,k} V_k) \, d\Gamma \, dt + \int_T \int_{\Gamma_q} \left(\frac{\partial g}{\partial q} - \theta^* \right) \\
& \times (Dq^0 - q_{,k} V_k) \, d\Gamma \, dt + \int_T \int_{\Gamma_u} \left(\frac{\partial g}{\partial u_i} - t_i^* \right) (Du_i^0 - u_{i,k} V_k) \\
& \times d\Gamma \, dt + \int_T \int_{\Gamma_t} \left(\frac{\partial g}{\partial t_i} + u_i^* \right) \left(Dt_i^0 - t_{i,k} V_k \right) d\Gamma \, dt \\
& + \int_\Omega \left[\dot{u}_i u_i^* \rho' - \left(\frac{\partial h}{\partial \theta} - \rho c \theta^* \right) (D\theta_0 - \theta_{,k} V_k) \right. \\
& - \left(\frac{\partial h}{\partial u_i} + \frac{\partial f}{\partial \dot{u}_i} + \rho \dot{u}_i^* \right) (Du_{i_0} - u_{i,k} V_k) - \left(\frac{\partial h}{\partial \dot{u}_i} - \rho u_i^* \right) \\
& \left. \times (D\dot{u}_{i_0} - \dot{u}_{i,k} V_k) \right]_{t=0} d\Omega - \int_\Omega [\dot{u}_i u_i^* \rho']_{t=t_f} \, d\Omega \\
& + \int_\Gamma [h - \rho \dot{u}_i u_i^*]_T V_n \, d\Gamma + \int_T \oint_\Lambda [\![g - q\theta^* + t_i u_i^*]\!] V_\mu \, d\Lambda \, dt
\end{aligned} \tag{53}$$

The following remarks can be made regarding the preceding expression:

1) Equation (53) represents the total variation of I corresponding to the simultaneous MLS variations. For a numerical evaluation of DI, solutions of both the primary and adjoint problems are needed, along with the deformation velocity V_i characterizing shape variations, and the local variations of all of the material and distributed load decision variables. The explicit forms of the total variations of the boundary and initial load decision variables also must be provided.

2) Although the boundary method of SA has been used, domain (i.e., volume) integrations are still needed, since the physical system under investigation is in dynamic response.[9] This is contrary to the case with steady-state structural optimization.

Example Problem

The SA expressions derived in the previous section may be validated by a simple one-dimensional example problem, which is amenable to analytical means. The material properties are taken as constant and uncontrollable. Only distributed load and shape variations will be considered through a finite number of decision parameters. A distributed heat source and a body force serve as the distributed loading functions. If a thin rod of infinitely small cross-sectional dimensions is considered as the physical one-dimensional domain, the heat source Q and the body force B may be adopted in the following forms:

$$Q = a_k \cos\zeta_0 x \quad \text{and} \quad B = bE \sin\zeta_0 x \tag{54}$$

where

$$\zeta_0 = \pi/2L \tag{55}$$

The decision parameters in the present problem will be taken as follows: the load parameters a and b and the shape parameter L. Thus, the present primary problem is given by

$0<x<L$:

$$\alpha(\theta_{xx} + a \cos\zeta_0 x) = \dot{\theta} \tag{56}$$

$$c_0^2(u_{xx} + b \sin\zeta_0 x - \beta\theta_x) = \ddot{u} \tag{57}$$

at $x = 0$:

$$\theta_x = u = 0 \tag{58}$$

at $x = L$:

$$\theta = u_x = 0 \tag{59}$$

at $t = 0$:

$$\theta = u = \dot{u} = 0 \tag{60}$$

where $\alpha = k/\rho c$ is the thermal diffusivity, c_0 the displacement wave speed, u the axial displacement in the rod, and subscript x refers to x differentiation. The rod is taken as thermally insulated, built-in at the end, and free with zero temperature at the other.

It may be seen from Eqs. (55-60) that the present thermoelastic problem admits a very simply solution, because of the adopted nature of Q and B, Eq. (54), and the homogeneity of the boundary and initial conditions, given in terms of the fundamental mode solution as

$$\theta(x,t) = \frac{a \cos\zeta_0 x}{\zeta_0^2}(1 - \exp[-\alpha\zeta_0^2 t]) \tag{61}$$

$$u(x,t) = \frac{\sin\zeta_0 x}{m_0\zeta_0^5}[(a\beta + b\zeta_0)m_0\zeta_0^2 - a\beta\omega_0^2 \exp[-\alpha\zeta_0^2 t] - a\beta\alpha\zeta_0^2\omega_0 \sin\omega_0 t - (a\beta\alpha^2\zeta_0 + bm_0)\zeta_0^3 \cos\omega_0 t] \tag{62}$$

where

$$m_0 = \alpha^2\zeta_0^2 + c_0^2 \quad \text{and} \quad \omega_0 = c_0\zeta_0 \tag{63}$$

Consider now a general performance criterion I given by

$$I = \int_0^{t_f}\int_0^L u \sin\zeta_0 x \, dx \, dt \tag{64}$$

It is noted that a special form has been adopted for I only for the sake of obtaining simple SA expressions derivable in terms of fundamental solutions. However, generalizations of I and the primary problem are straightforward. In particular, quadratic expressions of displacements may be considered in I in certain physical applications, for instance, in an "active" shape control problem of thermoelastic structures, where it may be desirable to nullify quasisteady types of shape distortions by using thermal and/or mechanical actuators.[1]

Using the adjoint variable method of SA, the adjoint heat flux and stress in the rod are simply given as

$$q^* = -k\theta_x^* \quad \text{and} \quad \sigma^* = Eu_x^* \tag{65}$$

From Eqs. (44) and (45), the preceding adjoint "constitutive" relations lead to the following adjoint problem corresponding to I:

$0<x<L$:

$$\alpha\left(\theta_{xx}^* + \frac{\beta E}{k}u_x^*\right) = -\dot{\theta}^* \tag{66}$$

$$c_0^2 u_{xx}^* + \frac{\sin\zeta_0 x}{\rho} = \ddot{u}^* \tag{67}$$

at $x = 0$:

$$\theta_x^* = u^* = 0 \tag{68}$$

at $x = L$:

$$\theta^* = u_x^* = 0 \tag{69}$$

at $t = t_f$:

$$\theta^* = u^* = \dot{u}^* = 0 \tag{70}$$

which yields a solution in the form given by

$$u^*(x,t) = \frac{\sin\zeta_0 x}{\rho\omega_0^2}[1 - \cos\omega_0(t_f - t)] \tag{71}$$

$$\theta^*(x,t) = \frac{\beta \cos\zeta_0 x}{km_0\zeta_0^5}\{m_0\zeta_0^2 - \omega_0^2 \exp[-\alpha\zeta_0^2(t_f - t)] - \alpha\zeta_0^2[\alpha\zeta_0^2 \cos\omega_0(t_f - t) + \omega_0 \sin\omega_0(t_f - t)]\} \tag{72}$$

The decision parameters in the present example problem are a, b, and L, where the rod is taken as fixed (i.e., unvaried) at $x = 0$, while it is varied in length at $x = L$. The total variation of I with respect to the variations of a, b, and L is given by the SA expression, Eq. (53), as

$$DI = \int_0^{t_f}\int_0^L \left(\theta^* Q' + u^* B' + \frac{\partial f}{\partial L}\delta L\right) dx \, dt + \int_0^{t_f} [(u \sin\zeta_0 x + Bu^* + k\theta_x\theta_x^* + \rho\dot{u}\dot{u}^*)V_n]_{x=L} \, dt \tag{73}$$

where the last term under the domain time integral comes from the fact that the integrand of I, i.e., $u \sin\zeta_0 x$, is an explicit function of L, since $\zeta_0 = \pi/2L$. In obtaining the preceding equation, the homogeneity of the boundary and time conditions of the primary and adjoint problems also have been used.

The local derivatives of Q and B may be obtained from Eq. (17) as

$$Q' = \frac{\partial Q}{\partial a}\,\delta a + \frac{\partial Q}{\partial L}\,\delta L \quad \text{and} \quad B' = \frac{\partial B}{\partial b}\,\delta b + \frac{\partial B}{\partial L}\,\delta L \tag{74}$$

which, by using Eqs. (54) and (53), yield

$$Q' = k\,\cos\zeta_0 x\,\delta a + \frac{2ak\,\zeta_0^2 x}{\pi}\,\delta L \tag{75}$$

$$B' = E\,\sin\zeta_0 x\,\delta b - \frac{2bE\zeta_0^2 x}{\pi}\cos\zeta_0 x\,\delta L \tag{76}$$

where δa, δb, and δL represent the variations in a, b, and L, respectively. The integrand f and its derivative with respect to L, $\partial f/\partial L$, also are given by

$$f = u\,\sin\zeta_0 x \quad \text{and} \quad \frac{\partial f}{\partial L} = -\frac{2\zeta_0^2 x}{\pi}\,u\,\cos\zeta_0 x \tag{77}$$

The normal component of V_i, V_n, is equal to $V_n = \delta L$ at $x = L$. It is noted that the DI expression, Eq. (73), does not require an assumption of the field distribution of the deformation velocity V_i in the rod, and that only its normal component V_n at the ends of the rod is needed.

Substituting Eqs. (61), (62), (71), (72), and (75–77) into Eq. (73), the total derivative (or the first variation) of I finally may be expressed in the following form:

$$\mathrm{D}I = \left.\frac{\delta I}{\delta a}\right|_{b,L}\delta a + \left.\frac{\delta I}{\delta b}\right|_{a,L}\delta b + \left.\frac{\delta I}{\delta L}\right|_{a,b}\delta L \tag{78}$$

where the sensitivity coefficients corresponding to δa, δb, and δL are given, respectively, by

$$\left.\frac{\delta I}{\delta a}\right|_{b,L} = \frac{\pi\beta}{4\alpha m_0\omega_0\zeta_0^{-8}}\{\alpha\zeta_0^{-4}m_0\omega_0 t_f + \omega_0^3(\exp[-\alpha\zeta_0^{-2}t_{f-1}]) - \alpha^2\zeta_0^{-4}[\alpha\zeta_0^{-2}\sin\omega_0 t_f + \omega_0(1-\cos\omega_0 t_f)]\} \tag{79}$$

$$\left.\frac{\delta I}{\delta b}\right|_{a,L} = \frac{\pi}{4\omega_0\zeta_0^{-3}}(\omega_0 t_f - \sin\omega_0 t_f) \tag{80}$$

$$\left.\frac{\delta I}{\delta L}\right|_{a,b} = \frac{a\beta}{2\alpha c_0 m_0^2\zeta_0^{-5}}\{4\alpha c_0 m_0^2\zeta_0^{-2}t_f - 6(\alpha^2\zeta_0^{-2} + 2c_0^2)\alpha^2 c_0\zeta_0^{-2} - 6c_0^5 + \alpha^2\zeta_0^{-3}[m_0 c_0^2 t_f - (5\alpha^2\zeta_0^{-2} + 3c_0^2)\alpha]\sin\omega_0 t_f + \alpha^2 c_0\zeta_0^{-2}[\alpha m_0\zeta_0^{-2}t_f + 2(3\alpha^2\zeta_0^{-2} + 2c_0^2)]\cos\omega_0 t_f + 2c_0^3[\alpha m_0\zeta_0^{-2}t_f + 4\alpha^2\zeta_0^{-2} + 3c_0^2]\exp[-\alpha\zeta_0^{-2}t_f]\} + \frac{b}{2c_0\zeta_0^{-3}}[(3 + \cos\omega_0 t_f - 4\sin\omega_0 t_f] \tag{81}$$

A simple check on the SA expressions, Eqs. (78–81), also is possible by first evaluating I, Eq. (64), as a function of a, b, and L. Thus, inserting the displacement solution, Eq. (62), into Eq. (64) and integrating, one may find that

$$I(a,b,L) = \frac{\pi a\beta}{4\alpha m_0\omega_0\zeta_0^{-8}}\{\alpha\zeta_0^{-4}m_0\omega_0 t_f + \omega_0^3(\exp[-\alpha\zeta_0^{-2}t_f]-1) - \alpha^2\zeta_0^{-4}[\alpha\zeta_0^{-2}\sin\omega_0 t_f + \omega_0(1-\cos\omega_0 t_f)]\} + \frac{\pi b}{4\omega_0\zeta_0^{-3}}(\omega_0 t_f - \sin\omega_0 t_f) \tag{82}$$

Now that I is given explicitly in terms of a, b, and L, direct (partial) differentiations of I may yield the same sensitivity coefficients as in Eqs. (79–81); i.e., it is found that

$$\frac{\partial I(a,b,L)}{\partial a} = \left.\frac{\delta I}{\delta a}\right|_{b,L} \tag{83}$$

$$\frac{\partial I(a,b,L)}{\partial b} = \left.\frac{\delta I}{\delta b}\right|_{a,L} \tag{84}$$

$$\frac{\partial I(a,b,L)}{\partial L} = \left.\frac{\delta I}{\delta L}\right|_{a,b} \tag{85}$$

which shows the validity of the SA expressions derived in the paper to be correct as applied to the present example problem.

Conclusions

Simultaneous material/load/shape variations of general performance criteria have been considered for thermoelastodynamic structures. Only the adjoint variable method of sensitivity analysis has been implemented, whereas the direct method[11] of evaluating the sensitivities of the primary (state) variables has not been investigated. As such, the present technique should be more advantageous for problems when the number of performance criteria is less than the number of decision parameters. The derived sensitivity expressions are valuable for the solution of the so-called inverse problems. In particular, physically realistic shape optimization and dynamic control of structures could be analyzed by the present analysis via mathematical programming methods after performing space-time discretizations, as the derivatives of performance criteria are already given by the sensitivity expressions.

A final point can be made such that dynamic shape control problems with time-dependent domains [i.e., moving boundary problems with $\Omega = \Omega(x_i,t)$][10] have not been considered in the present investigation. Such problems also may be important since, for example, one might wish to control in time (i.e., dynamically optimize) the surface shape of a space structure in order to realize a certain physical objective.

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